

# INVERSE PROBLEM FOR THE RIEMANNIAN WAVE EQUATION WITH DIRICHLET DATA AND NEUMANN DATA ON DISJOINT SETS

MATTI LASSAS AND LAURI OKSANEN

**ABSTRACT.** We consider the inverse problem to determine a smooth compact Riemannian manifold with boundary  $(M, g)$  from a restriction  $\Lambda_{\mathcal{S}, \mathcal{R}}$  of the Dirichlet-to-Neumann operator for the wave equation on the manifold. Here  $\mathcal{S}$  and  $\mathcal{R}$  are open sets in  $\partial M$  and the restriction  $\Lambda_{\mathcal{S}, \mathcal{R}}$  corresponds to the case where the Dirichlet data is supported on  $\mathbb{R}_+ \times \mathcal{S}$  and the Neumann data is measured on  $\mathbb{R}_+ \times \mathcal{R}$ . In the novel case where  $\overline{\mathcal{S}} \cap \overline{\mathcal{R}} = \emptyset$ , we show that  $\Lambda_{\mathcal{S}, \mathcal{R}}$  determines the manifold  $(M, g)$  uniquely, assuming that the wave equation is exactly controllable from the set of sources  $\mathcal{S}$ . Moreover, we show that the exact controllability can be replaced by the Hassell-Tao condition for eigenvalues and eigenfunctions, that is,

$$\lambda_j \leq C \|\partial_\nu \phi_j\|_{L^2(\mathcal{S})}^2, \quad j = 1, 2, \dots,$$

where  $\lambda_j$  are the Dirichlet eigenvalues and  $(\phi_j)_{j=1}^\infty$  is an orthonormal basis of the corresponding eigenfunctions.

## 1. INTRODUCTION

Let  $(M, g)$  be a smooth, connected and compact Riemannian manifold with boundary  $\partial M$ . We consider the wave equation with Dirichlet data  $f \in C_0^\infty((0, \infty) \times \partial M)$ ,

$$(1) \quad \begin{aligned} (\partial_t^2 - \Delta_g)u(t, x) &= 0, & \text{in } (0, \infty) \times M, \\ u|_{(0, \infty) \times \partial M} &= f, & \text{in } (0, \infty) \times \partial M, \\ u|_{t=0} = \partial_t u|_{t=0} &= 0, & \text{in } M, \end{aligned}$$

and denote by  $u^f = u(t, x)$  the solution of (1). For open and nonempty sets  $\mathcal{S}, \mathcal{R} \subset \partial M$  and  $T \in (0, \infty]$  we define the restricted Dirichlet-to-Neumann operator,

$$\Lambda_{M, g, \mathcal{S}, \mathcal{R}}^T : f \mapsto \partial_\nu u^f|_{(0, T) \times \mathcal{R}}, \quad f \in C_0^\infty((0, T) \times \mathcal{S}).$$

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Often we write  $\Lambda_{\mathcal{S},\mathcal{R}}^T = \Lambda_{M,g,\mathcal{S},\mathcal{R}}^T$  and  $\Lambda_{\mathcal{S},\mathcal{R}} = \Lambda_{\mathcal{S},\mathcal{R}}^\infty$ . When  $f$  is regarded as a boundary source, the operator  $\Lambda_{\mathcal{S},\mathcal{R}}^T$  models boundary measurements for the wave equation with sources producing the waves on  $(0, T) \times \mathcal{S}$  and the waves being observed on  $(0, T) \times \mathcal{R}$ . We consider the inverse boundary value problem to determine  $(M, g)$  from  $\Lambda_{M,g,\mathcal{S},\mathcal{R}}^T$ .

A problem of this type is often called a complete boundary data problem if  $\mathcal{S} = \mathcal{R} = \partial M$  and a partial boundary data problem if  $\mathcal{S} \neq \partial M$  or  $\mathcal{R} \neq \partial M$ . A sub-class of the partial boundary data problems are the local data problems where  $\mathcal{S} = \mathcal{R} \neq \partial M$ . The inverse problems with local data and the analogous partial boundary data problems with  $\mathcal{S} \cap \mathcal{R} \neq \emptyset$  have been studied broadly and we will give below a brief review of this literature. On the contrary, problems with  $\overline{\mathcal{S}} \cap \overline{\mathcal{R}} = \emptyset$ , that is, problems with disjoint partial data, have remained open to large extent. We are aware of only two previous results: in [40] Rakesh proved that the coefficients of a wave equation on a one-dimensional interval are determined by boundary measurements with sources supported on one end of the interval and the waves observed on the other end, and in [25] Imanuvilov, Uhlmann, and Yamamoto proved that the potential of a Schrödinger equation on a two-dimensional domain homeomorphic to a disc, where the boundary is partitioned into eight clockwise-ordered parts  $\Gamma_1, \Gamma_2, \dots, \Gamma_8$ , is determined by boundary measurements with sources supported on  $\mathcal{S} = \Gamma_2 \cup \Gamma_6$  and fields observed on  $\mathcal{R} = \Gamma_4 \cup \Gamma_8$ .

Inverse problems with partial boundary data are encountered in mathematical physics and in various applications. For example in medical imaging and in the geophysical imaging of the Earth, measurements can usually be done only a part of the boundary. Often it is not possible to observe fields on the same area where sources are controlled. For example in oil exploration, explosives are used as sources and hence it is difficult to measure waves near the sources. Also, many inverse scattering problems, such as the transmission problems on a line, are equivalent to disjoint partial data problems.

In this paper we consider the problem with  $\overline{\mathcal{S}} \cap \overline{\mathcal{R}} = \emptyset$  and show that the inverse problem to determine  $(M, g)$  given  $\Lambda_{\mathcal{S},\mathcal{R}}$  has unique solution if the wave equation (1) is exactly controllable from  $\mathcal{S}$ . We say that  $(M, g)$  is exactly controllable from  $\mathcal{S}$  in time  $T_0 > 0$  if the map

$$(2) \quad \begin{aligned} \mathcal{U} : L^2((0, T_0) \times \mathcal{S}) &\rightarrow L^2(M) \times H^{-1}(M), \\ \mathcal{U}(f) &= (u^f(T_0), \partial_t u^f(T_0)) \end{aligned}$$

is surjective. The condition by Bardos, Lebeau and Rauch gives a geometric characterization of exact controllability [5, 12]. In particular,

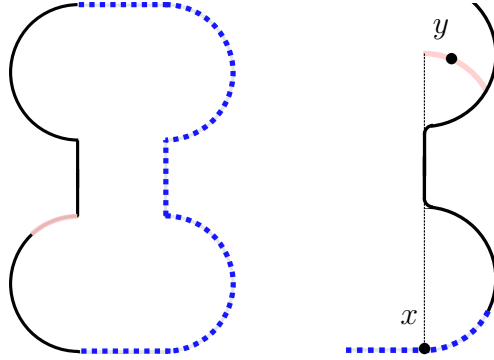


FIGURE 1. *On left*, the dogbone region with the Euclidean metric satisfies the assumptions of Theorem 1 when  $\mathcal{S}$  is the dashed blue part of the boundary and  $\mathcal{R}$  is the light red part of the boundary, see [5, Fig. 6]. *On right*, a detail of the dogbone region: for any point  $y$  on the light red arc, the point  $x$  is the closest point to  $y$  on the dashed blue part of the boundary. We overcome the difficulties arising from non-convexity by using modified boundary distance functions in Section 4.4.

if  $M$  has a strictly convex boundary, then exact controllability is valid when every geodesic, continued by normal reflection on the boundary and having length  $T_0$ , intersects  $\mathcal{S}$ . We refer to [5] for the precise formulation of the geometric condition in the case that  $\partial M$  is non-convex.

For our purposes the exact controllability can be replaced by a spectral condition that is strictly weaker in terms of the size of  $\mathcal{S}$ . Namely, let us denote the Dirichlet eigenvalues of  $-\Delta_g$  by

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \cdots \rightarrow \infty$$

and the corresponding  $L^2(M)$ -normalized eigenfunctions by  $\phi_j$ . That is,  $(\phi_j, \phi_k)_{L^2(M)} = \delta_{jk}$  and

$$-\Delta_g \phi_j = \lambda_j \phi_j \quad \text{on } M, \quad \phi_j|_{\partial M} = 0.$$

We say that the manifold  $(M, g)$  satisfies the Hassell-Tao condition for eigenvalues and eigenfunctions with the set  $\mathcal{S} \subset \partial M$  if there is  $C_0 > 0$  such that for all orthonormal bases  $(\phi_j)_{j=1}^\infty$  of eigenfunctions

$$(3) \quad \lambda_j \leq C_0 \|\partial_\nu \phi_j\|_{L^2(\mathcal{S})}^2, \quad \text{for all } j = 1, 2, \dots$$

We denote by  $d(x, y)$ ,  $x, y \in M$ , the Riemannian distance function of  $(M, g)$ . Our main result is the following.

**Theorem 1.** *Let  $(M_1, g_1)$  and  $(M_2, g_2)$  be  $C^\infty$ -smooth compact and connected Riemannian manifolds with boundary and let  $\mathcal{S}_j \subset \partial M_j$  and  $\mathcal{R}_j \subset \partial M_j$  be open non-empty sets with smooth boundaries for  $j = 1, 2$ . Suppose that there are diffeomorphisms  $\Phi : \overline{\mathcal{S}}_1 \rightarrow \overline{\mathcal{S}}_2$  and  $\Psi : \overline{\mathcal{R}}_1 \rightarrow \overline{\mathcal{R}}_2$  and that (H) or (H') holds, where*

(H)  $(M_j, g_j)$ ,  $j = 1, 2$ , is exactly controllable from  $\mathcal{S}_j$  in time  $T_0 > 0$  and there is  $T > T_0 + 2 \max_{j=1,2} \max_{x \in M_j} d(x, \mathcal{R}_j)$  such that

$$\Lambda_{M_1, g_1, \mathcal{S}_1, \mathcal{R}_1}^T f = \Psi^*(\Lambda_{M_2, g_2, \mathcal{S}_2, \mathcal{R}_2}^T(\Phi_* f))$$

for all  $f \in C_0^\infty((0, T) \times \mathcal{S}_1)$ .

(H')  $(M_j, g_j)$ ,  $j = 1, 2$ , satisfies the Hassell-Tao condition (3) with the set  $\mathcal{S}_j$  and

$$\Lambda_{M_1, g_1, \mathcal{S}_1, \mathcal{R}_1}^\infty f = \Psi^*(\Lambda_{M_2, g_2, \mathcal{S}_2, \mathcal{R}_2}^\infty(\Phi_* f))$$

for all  $f \in C_0^\infty((0, \infty) \times \mathcal{S}_1)$ .

Then  $(M_1, g_1)$  and  $(M_2, g_2)$  are isometric and there is such an isometry  $F : M_1 \rightarrow M_2$  that  $F|_{\mathcal{S}_1} = \Phi$  and  $F|_{\mathcal{R}_1} = \Psi$ .

Notice that  $M^{\text{int}}$  is not assumed to be known a priori and a part of the proof is to construct  $M$  as a smooth manifold. The same is true for  $\partial M \setminus \overline{\mathcal{S} \cup \mathcal{R}}$ .

While proving the above theorem, we develop a new geometric technique that we call modified boundary distance functions. This technique allows us to overcome difficulties arising from possible non-transversality between the geodesic flow and the boundary, see Figure 1. Such difficulties are present in various inverse problems, see e.g. [26, 42].

**1.1. Previous results.** The first results for inverse problems for the wave equation and the equivalent inverse problems the heat or the Schrödinger equations go back to the end of 50's when Krein studied the one-dimensional inverse problem for an inhomogeneous string,  $u_{tt} - c^2(x)u_{xx} = 0$ , see e.g. [30]. In his works, causality was transformed into analyticity of the Fourier transform of the solution. Later, in the late 60's, Blagovestchenskii [9, 10] developed a method to solve one dimensional inverse problems that was based on finite speed of wave propagation. In the late 80's Belishev [6] developed the boundary control method by combining the finite speed arguments with control theory to solve inverse problems for the wave equation in domains of the Euclidean space  $\mathbb{R}^n$ . A turning point in the study of inverse problems for wave equation happened in 1995 when Tataru proved a Holmgren-type uniqueness theorem for wave equations with non-analytic coefficients [45, 46]. This and the earlier results on the boundary control method by

Belishev and Kurylev [8] solved the inverse problem for the wave equation on a Riemannian manifold with complete data. For later development of the geometric boundary control method, see [2, 7, 27, 28, 31]. In all the above results measurements were assumed to be given either on the whole boundary or it was assumed that waves are observed on the same sets where sources are supported, that is,  $\mathcal{S} = \mathcal{R}$  in our notation. For the wave equation, the problem where the closures of the source domain  $\mathcal{S}$  and the observation domain  $\mathcal{R}$  do not intersect have been studied only in few papers, see [41] and the references therein, typically in the one-dimensional or radially symmetric cases. In [33] we studied the case when there are three disjoint domains  $\Sigma_1, \Sigma_2, \Sigma_3 \subset \partial M$  and assumed that all three Dirichlet-to-Neumann maps  $\Lambda_{\Sigma_1, \Sigma_2}$ ,  $\Lambda_{\Sigma_2, \Sigma_3}$ , and  $\Lambda_{\Sigma_3, \Sigma_1}$  are known.

The steady state solutions of the wave equation satisfy an elliptic equations and thus the inverse problems for elliptic equations can be in many cases considered as special cases of hyperbolic inverse problems with restricted data. A paradigm problem of this type is Calderón's inverse problem [13] that concerns the determination of the conductivity from the elliptic Dirichlet-to-Neumann map. A smooth isotropic conductivity in a bounded domain of  $\mathbb{R}^n$ ,  $n \geq 3$ , is determined by the elliptic Dirichlet-to-Neumann map as was shown in the seminal paper of Sylvester and Uhlmann [44]. In two dimensions the first unique identifiability result was proven in [38] for  $C^2$  conductivities and in [3] the problem was solved for  $L^\infty$  conductivities. The corresponding inverse problems for the elliptic Schrödinger equation has been solved in [11, 44]. For anisotropic smooth conductivity (corresponding to a general Riemannian metric) in  $\Omega \subset \mathbb{R}^n$ , Calderón's inverse problem was solved in two dimensions in [43] using the isotropic result [38]. The needed regularity was reduced to  $L^\infty$  later in [4]. In the case of dimension  $n \geq 3$ , Calderón's inverse problem is of geometrical nature and makes sense for general compact Riemannian manifolds with boundary, as was pointed out in [37]. This problem remains open, however, and we refer to [15, 20, 34] for partial results.

The partial data problem for the isotropic elliptic equation in  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , under certain geometrical restrictions on the sets  $\mathcal{S}$  and  $\mathcal{R}$ , has been solved by Kenig, Sjöstrand and Uhlmann [29]. Also, the local data problem for anisotropic elliptic equations in  $\Omega \subset \mathbb{R}^2$  was recently solved by Imanuvilov, Uhlmann and Yamamoto [23, 24]. For two-dimensional manifolds it was shown in [35] that the local boundary data for the Laplace-Beltrami equation with  $\mathcal{S} = \mathcal{R} \neq \partial M$  determines uniquely the manifold and the conformal class of the metric. Later the amount of needed measurements have been reduced in [19, 20,

21, 22]. For Riemannian surfaces the determination of the potential in a Riemann-Schrödinger equation with local data corresponding to  $\mathcal{S} = \mathcal{R} \neq \partial M$  has been solved in [16]. The above mentioned [25] is the only result with  $\mathcal{S} \cap \mathcal{R} = \emptyset$  concerning elliptic equations that we are aware of.

## 2. OUTLINE OF THE ARGUMENTS AND NOTATIONS

We begin by showing in Section 3 that the condition (H) in Theorem 1 implies the condition (H'). That is, we show that the Hassell-Tao condition (3) is implied by the exact controllability (2), and that the operator  $\Lambda_{\mathcal{S}, \mathcal{R}}^\infty$  is determined by the operator  $\Lambda_{\mathcal{S}, \mathcal{R}}^T$  when

$$(4) \quad T > T_0 + 2 \max_{x \in M} d(x, \mathcal{R})$$

and (1) is exactly controllable from  $\mathcal{S}$  in time  $T_0$ . If we take  $\mathcal{S} = \partial M$  and assume that  $M$  is non-trapping, then (3) was proved by Hassell and Tao [17] without using exact controllability. In the erratum for [17], it was noted that exact controllability yields (3) and this observation was attributed to Nicolas Burq. We give a short proof for this fact in Section 3 and show that (3) is strictly weaker than exact controllability in terms of the size of  $\mathcal{S}$ .

We will prove Theorem 1 in five steps that we will describe next. We denote  $SM := \{\xi \in TM; |\xi|_g = 1\}$ , that is,  $SM$  is the unit sphere bundle, and define  $\partial_\pm SM := \{\xi \in \partial SM; (\xi, \mp \nu)_g > 0\}$ , where  $\nu$  is the interior unit normal vector on  $\partial M$ . We define the *exit time* for  $(x, \xi) \in SM \setminus \partial_+ SM$ ,

$$\tau_M(x, \xi) := \inf\{s \in (0, \infty); \gamma(s; x, \xi) \in \partial M\},$$

where  $\gamma(\cdot; x, \xi)$  is the geodesic with the initial data  $\gamma(0) = x$ ,  $\dot{\gamma}(0) = \xi$ . Moreover, we define the strip

$$(5) \quad \begin{aligned} \mathcal{N}_{\mathcal{R}} &:= \{(s, y) \in (0, \infty) \times \mathcal{R}; s < \sigma_{\mathcal{R}}(y)\}, \\ \sigma_{\mathcal{R}}(y) &:= \max\{s \in (0, \tau_M(y, \nu)]; d(\gamma(s; y, \nu), \mathcal{R}) = s\}, \end{aligned}$$

and denote by  $M_{\mathcal{R}}$  the image of  $\mathcal{N}_{\mathcal{R}}$  under the map

$$(6) \quad (s, y) \mapsto \gamma(s; y, \nu).$$

The first step is to show that  $\Lambda_{\mathcal{S}, \mathcal{R}}$  determines  $(M_{\mathcal{R}}, g)$ . That is, we reconstruct a piece of  $(M, g)$  in the boundary normal coordinates. Notice that  $\sigma_{\mathcal{R}}(y) > 0$ , see e.g. [28, p. 50].

The second step is to show that  $\Lambda_{\mathcal{S}, \mathcal{R}}$  and  $(M_{\mathcal{R}}, g)$  determine the map

$$\Lambda_{\mathcal{S}, B} : f \mapsto u^f|_{(0, \infty) \times B}, \quad f \in C_0^\infty((0, \infty) \times \mathcal{S}),$$

where  $B \subset M_{\mathcal{R}}^{\text{int}}$  is a small ball sufficiently far away from the topological boundary of  $M_{\mathcal{R}}$ . The first two steps can be thought as bootstrapping steps that give us the interior data  $\Lambda_{\mathcal{S},B}$ . The remaining three steps employ only interior data, and we will reconstruct the unknown manifold  $(M, g)$  by extending the known subset iteratively.

We denote by  $x$  the center of  $B$  and define

$$\begin{aligned}\mathcal{N}_B &:= (0, \sigma^B) \times S_x M, \\ \sigma^B &:= \min_{\xi \in S_x M} \max\{s \in (0, \tau_M(x, \xi)]; d(\gamma(s; x, \xi), x) = s\}.\end{aligned}$$

We denote by  $M_B$  the image of  $\mathcal{N}_B$  under the map

$$(7) \quad (s, \xi) \mapsto \gamma(s; x, \xi).$$

The third step is to show that  $\Lambda_{\mathcal{S},B}$  determines  $(M_B, g)$ . That is, we reconstruct a piece of  $(M, g)$  in the geodesic normal coordinates. Essentially the same geometric method can be used to reconstruct  $(M_{\mathcal{R}}, g)$  and  $(M_B, g)$  from  $\Lambda_{\mathcal{S},\mathcal{R}}$  and  $\Lambda_{\mathcal{S},B}$ , respectively. We will describe this method in Section 4.

The fourth step is very similar with the second step. We show that  $\Lambda_{\mathcal{S},B}$  and  $(M_B, g)$  determine  $\Lambda_{\mathcal{S},B'}$  for a sufficiently small ball  $B' \subset M_B$ . The fifth step is to iterate the third and the fourth step and to glue the local reconstructions together. We will give the details of the second, fourth and fifth steps in Section 5.

### 3. EXACT CONTROLLABILITY AND TESTING WEAK CONVERGENCE OF SEQUENCES OF WAVES

Let us recall that for open  $\Gamma \subset \partial M$  and  $f \in C_0^\infty((0, \infty) \times \Gamma)$  we denote by  $u^f = u$  the solution of

$$\begin{aligned}\partial_t^2 u - \Delta_g u &= 0, & \text{in } (0, \infty) \times M, \\ u|_{(0, \infty) \times \partial M} &= f, & \text{in } (0, \infty) \times \partial M, \\ u|_{t=0} = \partial_t u|_{t=0} &= 0, & \text{in } M.\end{aligned}$$

We extend the notation  $u^f$  for open  $\Gamma \subset M^{\text{int}}$  and  $f \in C_0^\infty((0, \infty) \times \Gamma)$  as the solution  $u^f = u$  of

$$\begin{aligned}\partial_t^2 u - \Delta_g u &= f, & \text{in } (0, \infty) \times M, \\ u|_{(0, \infty) \times \partial M} &= 0, & \text{in } (0, \infty) \times \partial M, \\ u|_{t=0} = \partial_t u|_{t=0} &= 0, & \text{in } M.\end{aligned}$$

**Lemma 1** (Blagoveščenskiĭ's identity). *Let  $T > 0$  and let  $\Gamma$  be open either in  $\partial M$  or in  $M^{\text{int}}$ . Then for*

$$\psi \in C_0^\infty((0, \infty) \times \mathcal{S}), \quad f \in C_0^\infty((0, \infty) \times \Gamma).$$

we have

$$(8) \quad (u^f(T), u^\psi(T))_{L^2(M)} = (f, (J\Lambda_{\mathcal{S},\Gamma} - R\Lambda_{\mathcal{S},\Gamma}RJ)\psi)_{L^2((0,T)\times\Gamma)},$$

where  $R\psi(t) := \psi(T-t)$  and  $J\psi(t) := \frac{1}{2} \int_t^{2T-t} \psi(s)ds$ .

For a proof in the case  $\Gamma \subset \partial M$  see e.g. [28, Lem. 4.15]. The case  $\Gamma \subset M^{\text{int}}$  is analogous. In the lemma, the Riemannian volume measures are used on  $(M, g)$  and on  $(\partial M, g)$ . As we do not assume  $g|_{\partial M}$  to be known a priori, the right-hand side of (8) for  $\Gamma = \mathcal{R}$  is not trivially determined by the boundary measurement data  $\Lambda_{\mathcal{S},\mathcal{R}}$ . However, for our purposes it is sufficient to choose an arbitrary positive smooth measure  $d\tilde{S}$  on  $\mathcal{R}$ . Then there is strictly positive  $\mu \in C^\infty(\mathcal{R})$  such that

$$(9) \quad d\tilde{S} = \mu dS,$$

where  $dS$  is the Riemannian volume measure of  $(\mathcal{R}, g)$ . Moreover,

$$\begin{aligned} (f, K\psi)_{L^2((0,T)\times\mathcal{R}; dt \otimes d\tilde{S})} &= (\mu f, K\psi)_{L^2((0,T)\times\mathcal{R})} \\ &= (u^{\mu f}(T), u^\psi(T))_{L^2(M)}, \end{aligned}$$

where  $K := J\Lambda_{\mathcal{S},\mathcal{R}} - R\Lambda_{\mathcal{S},\mathcal{R}}RJ$ .

Let us assume for a moment that the exact controllability (2) holds and let  $(f_j)_{j=1}^\infty \subset C_0^\infty((0, \infty) \times \mathcal{R})$ . Then the functions  $u^{f_j}(T_0)$  tend weakly to zero in  $L^2(M)$  as  $j \rightarrow \infty$  if and only if the inner products (8) with  $f = f_j$  and  $T = T_0$  tend to zero for all  $\psi \in L^2((0, T_0) \times \mathcal{S})$ . In this section we will describe a method to determine if  $(u^{\mu f_j}(T))_{j=1}^\infty$  is weakly convergent under the weaker assumption (3). Let us point out that the assumption (3) is needed only in this step.

If  $\mathcal{R} = \mathcal{S}$  then Lemma 1 allows us to compute  $\|u^{\mu f_j}(T)\|_{L^2(M)}$  and we can easily determine if  $(u^{\mu f_j}(T))_{j=1}^\infty$  tends to zero. In this case the assumption (3) is not needed since we can replace the exact controllability by the approximate controllability, the latter of which holds for arbitrary  $(M, g)$  by Tataru's unique continuation [45], see e.g. [28, Th. 3.10]. The case  $\mathcal{R} = \mathcal{S}$  was solved originally in [27].

**3.1. Exact controllability and the Hassell-Tao condition.** We recall that the eigenvalues of the positive Laplace-Beltrami operator  $-\Delta_g$  with the domain  $H^2(M) \cap H_0^1(M)$  are denoted by  $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \cdots \rightarrow \infty$  and the corresponding  $L^2(M)$ -normalized eigenfunctions by  $\phi_j$ .

It is well-known, see e.g. [5], that the exact controllability (2) is equivalent with the continuous observability inequality,

$$(10) \quad \|(w_0, w_1)\|_{H_0^1(M) \times L^2(M)} \leq C \|\partial_\nu w\|_{L^2((0, T_0) \times \mathcal{S})},$$



where  $w$  is the solution of the wave equation

$$(11) \quad \begin{aligned} \partial_t^2 w - \Delta_g w &= 0, & \text{in } (0, T_0) \times M, \\ w|_{(0, T_0) \times \partial M} &= 0, & \text{in } (0, T_0) \times \partial M, \\ w|_{t=0} &= w_0, \quad \partial_t w|_{t=0} = w_1 & \text{in } M. \end{aligned}$$

**Lemma 2.** *Suppose that (10) holds. Then there is  $C > 0$  such that*

$$(12) \quad \sqrt{\lambda_j + 1} \leq C \|\phi_j\|_{L^2(\mathcal{S})}, \quad \text{for all } j = 1, 2, \dots$$

*Proof.* Notice that if (10) holds then it holds also when  $T_0$  is replaced by a larger time. Let  $w_0 = \phi_j$  and  $w_1 = 0$  in (11). By writing the corresponding solution  $w$  in the eigenbasis  $(\phi_j)_{j=1}^\infty$  we see that

$$(13) \quad w(t, x) = \cos(\sqrt{\lambda_j} t) \phi_j(x).$$

Moreover,

$$\begin{aligned} \|(w_0, w_1)\|_{H_0^1(M) \times L^2(M)}^2 &= \|\phi_j\|_{H_0^1(M)}^2 = (d\phi_j, d\phi_j)_{L^2(M)} + 1 \\ &= (\Delta_g \phi_j, \phi_j)_{L^2(M)} + 1 = \lambda_j + 1. \end{aligned}$$

If  $T > \frac{1}{2\sqrt{\lambda_1}}$  then

$$\int_0^T \cos^2(\sqrt{\lambda_j} t) dt = \frac{T}{2} + \frac{\sin(2\sqrt{\lambda_j} T)}{4\sqrt{\lambda_j}} \geq \frac{T}{2} - \frac{1}{4\sqrt{\lambda_1}} > 0.$$

If also  $T \geq T_0$ , then we have by (13) and (10) that

$$\begin{aligned} \lambda_j + 1 &= \|(w_0, w_1)\|_{H_0^1(M) \times L^2(M)}^2 \leq C \|\partial_\nu w\|_{L^2((0, T) \times \Gamma)}^2 \\ &\leq C' \frac{\|\partial_\nu w\|_{L^2((0, T) \times \Gamma)}^2}{\int_0^T \cos^2(\sqrt{\lambda_j} t) dt} = C' \int_\Gamma (\partial_\nu \phi_j)^2 dS. \end{aligned}$$

□

**Example 1.** *Let  $(M, g)$  be the Euclidean unit disc in  $\mathbb{R}^2$  and denote*

$$\Gamma_s := \{e^{i2\pi\theta}; \theta \in (0, s)\}.$$

*Exact controllability from  $\Gamma_s$  holds for  $s > 1/2$  and does not hold for  $s < 1/2$ , see e.g. [5]. However, the condition (12) holds on  $\mathcal{S} = \Gamma_s$  for  $s \geq 1/4$ .*

*Proof.* An orthonormal basis of eigenfunctions can be chosen in the polar coordinates  $(r, \theta) \in (0, 1] \times (0, 2\pi]$  as

$$\phi_{mn1} = c_{nm} \cos(m\theta) J_m(z_{mn} r), \quad \phi_{mn2} = c_{nm} \sin(m\theta) J_m(z_{mn} r),$$

where  $c_{nm}$  is a normalization constant and  $z_{mn}$  is the  $n$ th positive zero of the  $m$ th Bessel function  $J_m$ . The corresponding eigenvalues are

$\lambda_{nmj} = z_{mn}^2$  and we have used the indices  $m = 0, 1, 2, \dots$ ,  $n = 1, 2, \dots$  and  $j = 1, 2$ . Notice that  $z_{mn} \neq z_{m'n'}$  if  $(m, n) \neq (m', n')$ , see e.g. [47, p. 484]. Thus any normalized eigenfunction  $\phi$  corresponding to the eigenvalue  $z_{mn}^2$  can be written as  $\phi = a\phi_{mn1} + b\phi_{mn2}$  with  $a^2 + b^2 = 1$ . Moreover, for  $m > 0$ ,

$$\|\partial_\nu \phi\|_{L^2(\Gamma_s)}^2 = c_{nm}^2 z_{mn}^2 (J'_m(z_{mn}))^2 \int_0^{2\pi s} (a \cos(m\theta) + b \sin(m\theta))^2 d\theta.$$

In particular,

$$\begin{aligned} \int_0^{\pi/2} (a \cos(m\theta) + b \sin(m\theta))^2 d\theta &= \frac{-2(-1 + (-1)^m)ab + m\pi}{4m} \\ &\geq \frac{m\pi - 2}{4m} > \frac{\pi}{12} = \frac{1}{12} \int_0^{2\pi} (a \cos(m\theta) + b \sin(m\theta))^2 d\theta. \end{aligned}$$

Thus the lower bound (12) holds for  $\Gamma_{1/4}$  with the constant  $C/12$  where  $C$  is the corresponding constant for  $\Gamma_1$ .  $\square$

### 3.2. Testing weak convergence of sequences of waves.

**Lemma 3.** *Let  $\Gamma \subset \partial M$  be open and nonempty and let*

$$T > \max_{x \in M} d(x, \Gamma).$$

*A sequence  $(v_l)_{l=1}^\infty \subset L^2(M)$  converges to zero weakly in  $L^2(M)$  if and only if both (a) and (b) hold, where*

- (a) *For all sequences  $(\psi_m)_{m=1}^\infty \subset C_0^\infty((0, T) \times \Gamma)$  such that the sequence  $(u^{\psi_m}(T))_{m=1}^\infty \subset L^2(M)$  is bounded, there is  $C > 0$  satisfying*

$$|(v_l, u^{\psi_m}(T))_{L^2(M)}| \leq C \quad \text{for all } l, m = 1, 2, \dots$$

- (b)  *$\lim_{l \rightarrow \infty} (v_l, u^\psi(T))_{L^2(M)} = 0$  for all  $\psi \in C_0^\infty((0, T) \times \Gamma)$ .*

*Proof.* In the proof we omit writing  $L^2(M)$  as a subscript. If  $(v_l)_{l=1}^\infty$  is weakly convergent to zero then (b) holds trivially and (a) holds since weakly convergent sequences are bounded.

Let us assume (a) and (b). We will first show that  $(v_l)_{l=1}^\infty$  is bounded in  $L^2(M)$ . Let  $v_0 \in L^2(M)$ . Tataru's unique continuation [45] implies approximate controllability, see e.g. [28, Th. 3.10]. Thus there is  $(\psi_m)_{m=1}^\infty \subset C_0^\infty((0, T) \times \Gamma)$  such that  $u^{\psi_m}(T) \rightarrow v_0$  in  $L^2(M)$ . Hence for all  $l$

$$|(v_l, v_0)| = \lim_{m \rightarrow \infty} |(v_l, u^{\psi_m}(T))| \leq C,$$

where  $C > 0$  is independent of  $l$ . Thus  $(v_l)_{l=1}^\infty$  is weakly bounded, and hence bounded in the norm.

Let  $\epsilon > 0$  and fix  $m$  such that  $\|v_0 - u^{\psi_m}(T)\| \leq \epsilon$ . Then for large  $l$

$$\begin{aligned} |(v_l, v_0)| &\leq \sup_l \|v_l\| \|v_0 - u^{\psi_m}(T)\| + (v_l, u^{\psi_m}(T)) \\ &\leq \sup_l \|v_l\| \epsilon + \epsilon. \end{aligned}$$

As  $\epsilon > 0$  and  $v_0 \in L^2(M)$  are arbitrary,  $(v_l)_{l=1}^\infty$  converges weakly to zero.  $\square$

Let us reindex the orthonormal basis  $(\phi_j)_{j=1}^\infty$  of Dirichlet eigenfunctions so that for  $j = 1, 2, \dots$ , the functions

$$\phi_{jk}, \quad k = 1, 2, \dots, K_j$$

span the space of eigenfunctions corresponding to the eigenvalue  $\lambda_j$ . Here  $0 < \lambda_1 < \lambda_2 < \lambda_3 \dots \rightarrow \infty$  and  $K_j$  is the multiplicity of  $\lambda_j$ . Let us choose a positive smooth measure  $d\tilde{S}$  on  $\bar{\mathcal{S}}$ . Then there is a strictly positive function  $\mu \in C^\infty(\bar{\mathcal{S}})$  such that (9) holds. As explained in [33, pp. 5-6] the Fourier transform of the operator  $\Lambda_{\mathcal{S}, \mathcal{R}}$  with respect to the time variable is a meromorphic map, and its poles and residues determine the Dirichlet eigenvalues  $\lambda_j$  and also the spaces

$$E_j := \text{span}\{\mu^{-1}\partial_\nu\phi_{jk}|_{\mathcal{S}}; \quad k = 1, 2, \dots, K_j\} \subset C^\infty(\bar{\mathcal{S}}),$$

for each  $j = 1, 2, \dots$ .

**Proposition 1.** *Let  $T > 0$  and  $(\psi_m)_{m=1}^\infty \subset C_0^\infty((0, \infty) \times \mathcal{S})$  and suppose that there is  $C_0 > 0$  such that*

$$\sqrt{\lambda_j} \leq C_0 \|\partial_\nu\phi_{jk}\|_{L^2(\mathcal{S})}, \quad k = 1, \dots, K_j, \quad j = 1, 2, \dots$$

*Then the following are equivalent*

- (i) *The sequence  $u^{\psi_m}(T)$ ,  $m = 1, 2, \dots$ , is bounded in  $L^2(M)$ .*
- (ii) *For all  $C_1 > 0$  and  $(e_j)_{j=1}^\infty \subset C^\infty(\mathcal{S})$  satisfying*

$$(14) \quad e_j \in E_j, \quad \|e_j\|_{L^2(\mathcal{S}; d\tilde{S})} \leq C_1 \sqrt{\lambda_j},$$

*there is  $C_2 > 0$  such that*

$$\sup_m \sum_{j=1}^\infty \left( \int_0^T s_j(t) \int_{\mathcal{S}} \psi_m(t, x) e_j(x) d\tilde{S}(x) dt \right)^2 \leq C_2,$$

$$\text{where } s_j(t) := \sin(\sqrt{\lambda_j}(T - t)) / \sqrt{\lambda_j}.$$

*Proof.* Let (i) hold and let  $e_j$ ,  $j = 1, 2, \dots$ , satisfy (14). As the choice of the orthonormal basis  $\{\phi_{jk}; k = 1, 2, \dots, K_j\}$  in the  $j$ th eigenspace is unique only up to a rotation, we may assume after a rotation that

$$\mu e_j = c_j \partial_\nu \phi_{j1}|_{\mathcal{S}}, \quad j = 1, 2, \dots,$$

where  $c_j > 0$  are some constants. Moreover, the sequence  $(c_j)_{j=1}^\infty$  is bounded. Indeed,

$$c_j \frac{\sqrt{\lambda_j}}{C_0} \leq \|c_j \partial_\nu \phi_{j1}\|_{L^2(\mathcal{S})} = \|\mu e_j\|_{L^2(\mathcal{S})} \leq C \|e_j\|_{L^2(\mathcal{S}; d\tilde{S})} \leq CC_1 \sqrt{\lambda_j}.$$

Denoting  $C' := (CC_0C_1)^2$  we have

$$\begin{aligned} \sum_{j=1}^\infty \left( \int_0^T s_j \int_{\mathcal{S}} \psi_m e_j d\tilde{S} dt \right)^2 &= \sum_{j=1}^\infty c_j^2 \left( \int_0^T s_j \int_{\mathcal{S}} \psi_m \partial_\nu \phi_{j1} dS dt \right)^2 \\ &\leq C' \sum_{j=1}^\infty (u^{\psi_m}(T), \phi_{j1})_{L^2(M)}^2 \leq C' \|u^{\psi_m}(T)\|_{L^2(M)}^2, \end{aligned}$$

and (ii) holds.

Let us now assume (ii) and let  $v \in L^2(M)$ . We denote by  $P_j$  the orthogonal projection onto the  $j$ th eigenspace. We may rotate again the basis  $\{\phi_{jk}; k = 1, 2, \dots, K_j\}$  so that

$$\phi_{j1} = \frac{P_j v}{\|P_j v\|_{L^2(M)}}, \quad \text{for all } j \text{ satisfying } P_j v \neq 0.$$

Then  $(v, \phi_{jk}) = 0$  for all  $k \geq 2$  and all  $j$ .

We may choose  $e_j := \mu^{-1} \partial_\nu \phi_{j1}|_{\mathcal{S}}$  in (ii). Indeed,

$$\|\mu^{-1} \partial_\nu \phi_{j1}\|_{L^2(\mathcal{S}; d\tilde{S})} \leq C \|\partial_\nu \phi_{j1}\|_{L^2(\partial M)} \leq C_1 \sqrt{\lambda_j},$$

where the second inequality holds by [17]. We have

$$\begin{aligned} |(v, u^{\psi_m}(T))|^2 &= \left| \sum_{j=1}^\infty (v, \phi_{j1}) \int_0^T s_j \int_{\mathcal{S}} \psi_m \partial_\nu \phi_{j1} dS dt \right|^2 \\ &\leq \sum_{j=1}^\infty (v, \phi_{j1})^2 \sum_{j=1}^\infty \left( \int_0^T s_j \int_{\mathcal{S}} \psi_m e_j \mu dS dt \right)^2 \\ &\leq \|v\|_{L^2(M)}^2 \sup_m \sum_{j=1}^\infty \left( \int_0^T s_j \int_{\mathcal{S}} \psi_m e_j d\tilde{S} dt \right)^2 \end{aligned}$$

and the sequence  $u^{\psi_m}(T)$ ,  $m = 1, 2, \dots$ , is weakly bounded.  $\square$

### 3.3. Continuation of the data in time.

**Lemma 4.** *Let  $\mathcal{S}, \mathcal{R} \subset \partial M$  be open and non-empty and suppose that (1) is exactly controllable from  $\mathcal{S}$  in time  $T_0$ . If  $T$  satisfies (4) then  $\Lambda_{\mathcal{S}, \mathcal{R}}^T$  determines  $\Lambda_{\mathcal{S}, \mathcal{R}}^\infty$ .*

*Proof.* Let  $\delta \in (0, T_0)$  satisfy

$$T > T_0 + 2 \max_{x \in M} d(x, \mathcal{R}) + \delta,$$

and let  $f \in C_0^\infty((0, T + \delta) \times \mathcal{S})$ . We choose  $h \in C_0^\infty((0, 2\delta) \times \mathcal{S})$  and  $h' \in C_0^\infty((\delta, T + \delta) \times \mathcal{S})$  such that  $f = h + h'$ . As the coefficients of (1) are time independent, we may translate in time and see that  $\Lambda_{\mathcal{S}, \mathcal{R}} h'(t)$ ,  $t \in (\delta, T + \delta)$ , is determined by  $\Lambda_{\mathcal{S}, \mathcal{R}}^T$ .

We recall that  $\Lambda_{\mathcal{S}, \mathcal{R}}^T : L^2((0, T) \times \mathcal{S}) \rightarrow H^{-1}((0, T) \times \mathcal{R})$  is continuous [32]. Let us suppose that  $\tilde{h} \in L^2((\delta, T_0 + \delta) \times \mathcal{S})$  satisfies

$$(15) \quad \Lambda_{\mathcal{S}, \mathcal{R}}^T h(t) = \Lambda_{\mathcal{S}, \mathcal{R}}^T \tilde{h}(t), \quad \text{for } t \in (T_0 + \delta, T).$$

Notice that such a function  $\tilde{h}$  exists. Indeed, by exact controllability (2) there is  $\tilde{h} \in L^2((\delta, T_0 + \delta) \times \mathcal{S})$  such that

$$(u^h(T_0 + \delta), \partial_t u^h(T_0 + \delta)) = (u^{\tilde{h}}(T_0 + \delta), \partial_t u^{\tilde{h}}(T_0 + \delta)).$$

As also  $h(t) = \tilde{h}(t) = 0$  for  $t > T_0 + \delta$ , we have  $u^h(t) = u^{\tilde{h}}(t)$  for  $t > T_0 + \delta$ . We have shown that there is  $\tilde{h} \in L^2((\delta, T_0 + \delta) \times \mathcal{S})$  satisfying (15).

By (15) the Cauchy data of  $u^h$  and  $u^{\tilde{h}}$  coincide on  $(T_0 + \delta, T) \times \mathcal{R}$ . Thus Tataru's unique continuation [45] implies that  $u^h(t + T_0 + \delta) = u^{\tilde{h}}(t + T_0 + \delta)$  for  $t$  near  $(T - T_0 - \delta)/2$ , see e.g. [28, Th. 3.10]. In particular,  $u^h(t) = u^{\tilde{h}}(t)$  for  $t > T - \max_{x \in M} d(x, \mathcal{R})$ . Analogously to the above case of  $\Lambda_{\mathcal{S}, \mathcal{R}} h'(t)$ , we see that  $\Lambda_{\mathcal{S}, \mathcal{R}} h(t) = \Lambda_{\mathcal{S}, \mathcal{R}} \tilde{h}(t)$ ,  $t \in (T, T + \delta)$ , is determined by  $\Lambda_{\mathcal{S}, \mathcal{R}}^T$ . In particular, we have determined

$$\Lambda_{\mathcal{S}, \mathcal{R}} f(t) = \Lambda_{\mathcal{S}, \mathcal{R}} h(t) + \Lambda_{\mathcal{S}, \mathcal{R}} h'(t), \quad t \in (T, T + \delta).$$

That is, we have shown that  $\Lambda_{\mathcal{S}, \mathcal{R}}^T$  determines  $\Lambda_{\mathcal{S}, \mathcal{R}}^{T+\delta}$ . By iterating the above argument we see that  $\Lambda_{\mathcal{S}, \mathcal{R}}^\infty$  is determined.  $\square$

#### 4. LOCAL RECONSTRUCTION OF THE MANIFOLD

In this section we describe a method to reconstruct  $(M_{\mathcal{R}}, g)$  and  $(M_B, g)$  from  $\Lambda_{\mathcal{S}, \mathcal{R}}$  and  $\Lambda_{\mathcal{S}, B}$ , respectively. We will first consider the local reconstruction method under the additional assumption that the functions  $\sigma_{\mathcal{R}}$  and  $\sigma^B$  are known, and then show how these functions can be reconstructed from the data  $\Lambda_{\mathcal{S}, \mathcal{R}}$  and  $\Lambda_{\mathcal{S}, B}$ , respectively. Let us recall that  $\sigma_{\mathcal{R}}(y)$  indicates the distance when the normal geodesic starting from  $y \in \mathcal{R}$  hits to the boundary or to a point on the cut locus, see (5). The main difficulty when reconstructing  $\sigma_{\mathcal{R}}(y)$  is that the normal geodesic may intersect  $\partial M$  tangentially and this is hard

to detect from the data, see Figure 1. To deal with this difficulty we present a method that is based on “perturbing” the boundary  $\partial M$ .

We will next give a series of lemmas that is common for the reconstruction method of  $\sigma_{\mathcal{R}}$  and  $\sigma^B$  and that of  $(M_{\mathcal{R}}, g)$  and  $(M_B, g)$ . We start by introducing the modified distance function  $d_h$ . Let  $\Gamma \subset M$  and  $h : \Gamma \rightarrow \mathbb{R}$ . We define

$$\begin{aligned} d_h(x, y) &:= d(x, y) - h(y), \quad x \in M, \ y \in \Gamma, \\ d_h(x, \Gamma) &:= \inf_{y \in \Gamma} d_h(x, y), \quad x \in M, \end{aligned}$$

where  $d$  is the Riemannian distance function of  $(M, g)$ . Moreover, we define the modified domain of influence

$$M(\Gamma, h) := \{x \in M; \ d_h(x, \Gamma) \leq 0\},$$

and denote for  $T > 0$

$$\mathcal{B}(\Gamma, h; T) := \{(t, y) \in (0, T) \times \Gamma; \ T - h(y) < t\}.$$

To simplify the notation, we define  $M(\Gamma, r)$  for a constant  $r \in (0, \infty)$  by  $M(\Gamma, h)$  where  $h(y) = r$ ,  $y \in \Gamma$ . Notice that if  $h \in C(\bar{\Gamma})$  then

$$M(\Gamma, h) = \{x \in M; \text{ there is } y \in \bar{\Gamma} \text{ such that } d(x, y) \leq h(y)\},$$

and our definition coincides with the definition of the domain of influence in [39]. In particular, for  $\Gamma = \{y\}$  the set  $M(\Gamma, h)$  is the closed geodesic ball with radius  $h(y)$ . In this case, we denote also  $M(y, h) := M(\Gamma, h)$ .

We will show first that  $\Lambda_{\mathcal{S}, \mathcal{R}}$  and  $\Lambda_{\mathcal{S}, B}$  determine certain relations between domains of influences, and then that these relations determine  $(M_{\mathcal{R}}, g)$  and  $(M_B, g)$ . The latter step is purely geometric.

**4.1. From weakly convergent sequences of waves to relations between domains of influences.** Tataru’s unique continuation result [45] implies that the wave equation (1) is approximately controllable, that is, we have the following lemma.

**Lemma 5.** *Let  $T > 0$  and suppose that  $\Gamma$  is open either in  $\partial M$  or in  $M^{\text{int}}$  and that  $h \in C(\bar{\Gamma})$  satisfies  $h \leq T$  pointwise. In the case when  $\Gamma \subset M^{\text{int}}$  suppose, moreover, that  $h > 0$  pointwise. Then*

$$(16) \quad \{u^f(T); \ f \in C_0^\infty(\mathcal{B}(\Gamma, h; T))\}$$

*is dense in*

$$L^2(M(\Gamma, h)) := \{v \in L^2(M); \ \text{supp}(v) \subset M(\Gamma, h)\}.$$

Notice that the finite speed of propagation implies that  $u^f(T)$  is supported in  $M(\Gamma, h)$  if  $f$  is supported in  $\mathcal{B}(\Gamma, h; T)$ . In the literature Lemma 5 is usually proved only in the case of a constant function  $h$ , see e.g. [28, Th. 3.10]. However, the case  $h \in C(\bar{\Gamma})$  can be reduced to this case by approximating  $h$  with piecewise constant functions, see [39, Lemmas 4.2 and 4.3].

**Lemma 6.** *Let  $T > 0$ ,  $J \in \mathbb{N}$ ,  $j = 1, \dots, J$  and let  $\Gamma_j$  be open either in  $\partial M$  or in  $M^{\text{int}}$ . Let  $h_j \in C(\bar{\Gamma}_j)$  satisfy  $h_j \leq T$  pointwise and, in the case  $\Gamma_j \subset M^{\text{int}}$ , also  $h_j > 0$ . We define  $\Gamma := \bigcup_{j=1}^J \Gamma_j$ ,*

$$(17) \quad h(y) := \max\{h_j(y); j \text{ satisfies } \bar{\Gamma}_j \ni y\}$$

*and denote  $\mathcal{U}_1(f) := u^f(T)$ . Let  $\Gamma_0$  be open either in  $\partial M$  or in  $M^{\text{int}}$  and let  $s_0 \in (0, T]$ . Let  $\mu \in C^\infty(\Gamma \cup \Gamma_0)$  be strictly positive. Then the following properties are equivalent:*

- (i)  $M(\Gamma_0, s_0) \subset \bigcup_{j=1}^J M(\Gamma_j, h_j)$ .
- (ii) *For all  $f_0 \in C_0^\infty(\mathcal{B}(\Gamma_0, s_0; T))$  there is  $(f_j)_{j=1}^\infty \subset C_0^\infty(\mathcal{B}(\Gamma, h; T))$  such that  $(\mathcal{U}_1(\mu(f_0 - f_j)))_{j=1}^\infty$  tends to zero weakly in  $L^2(M)$ .*

*Proof.* Notice that  $f \mapsto \mu f$  is a bijection on  $C_0^\infty(\mathcal{B}(\Gamma_0, s_0; T))$  and also on  $C_0^\infty(\mathcal{B}(\Gamma, h; T))$ . Thus we lose no generality by assuming that  $\mu = 1$  identically. We have

$$\begin{aligned} M(\Gamma, h) &= \{x \in M; \text{ there is } y \in \bar{\Gamma} \text{ s.t. } d(x, y) \leq h(y)\} \\ &= \bigcup_{j=1}^J M(\Gamma_j, h_j). \end{aligned}$$

The implication from (i) to (ii) follows from Lemma 5. We will now show that (ii) implies (i). We denote

$$\begin{aligned} M_0 &:= M(\Gamma_0, s_0), \quad M_1 := M(\Gamma, h), \\ S_0 &:= \mathcal{B}(\Gamma_0, s_0; T), \quad S_1 := \mathcal{B}(\Gamma, h; T). \end{aligned}$$

Let us assume that (i) does not hold and let  $x \in M_0 \setminus M_1$ . As  $M_1$  is closed, there is a neighborhood  $U$  of  $x$  such that  $U \cap M_1 = \emptyset$ . We will show next that  $U \cap M_0^{\text{int}}$  is nonempty.

If  $x \in \bar{\Gamma}_0$ , then points close to  $x$  are in  $M_0$  since  $s_0 > 0$ . Let us now assume that  $x \notin \bar{\Gamma}_0$ . Then there is a path  $\gamma$  from  $x$  to a point  $y_0 \in \bar{\Gamma}_0$  such that its length satisfies  $0 < l(\gamma) \leq s_0$ . We may assume that  $\gamma$  is a shortest path between  $x$  and  $y_0$  and that it has unit speed [1]. Then  $\gamma(t) \in U \cap M_0^{\text{int}}$  for small  $t > 0$ .

We have shown that  $U \cap M_0^{\text{int}}$  is nonempty. Hence there is a nonempty open  $V \subset M_0$  such that  $V \cap M_1 = \emptyset$ . By Lemma 5 there is a smooth

function  $f_0$  supported in  $S_0$  such that  $\int_V u^{f_0}(T)dx \neq 0$ . However, by finite speed of propagation  $u^f(T)|_V = 0$  for any  $f$  supported in  $S_1$ . Thus

$$(u^{f_0}(T) - u^f(T), 1_V) = (u^{f_0}(T), 1_V) \neq 0,$$

for all  $f$  supported in  $S_1$  and (ii) does not hold.  $\square$

**Proposition 2.** *Let  $\mathcal{S}, \mathcal{R} \subset \partial M$  and  $B \subset M^{\text{int}}$  be open, non-empty sets with smooth boundaries and suppose that the Hassell-Tao condition (3) holds with the set  $\mathcal{S}$ . Then  $\Lambda_{\mathcal{S}, \mathcal{R}}$  together with the smooth structure of  $\overline{\mathcal{S}} \cup \overline{\mathcal{R}}$  determines the relation*

(18)

$$\{(y_0, y_1, r, s, t) \in \mathcal{R}^2 \times (0, \infty)^3; M(y_0, r) \subset M(\mathcal{R}, s) \cup M(y_1, t)\}.$$

Moreover,  $\Lambda_{\mathcal{S}, B}$  together with the smooth structure of  $\overline{\mathcal{S}} \cup \overline{B}$  determines the relation

(19)

$$\{(y_0, y_1, r, s, t) \in \partial B^2 \times (0, \infty)^3; M(y_0, r) \subset M(B, s) \cup M(y_1, t)\}.$$

*Proof.* Notice that  $M(\Gamma, r) = M = M(\Gamma, T)$  for  $r \geq T$  and non-empty  $\Gamma \subset M$  since  $T \geq \text{diam}(M)$ . The claim follows from Lemmas 1, 3 and 6, Proposition 1 and the following observation. Let  $y_0, y_1 \in M$ ,  $s_0, s_1 > 0$  and  $J \in \mathbb{N}$ . Let  $\Gamma_j$  be open either in  $\partial M$  or in  $M^{\text{int}}$  and  $h_j \in C(\overline{\Gamma}_j)$  for  $j = 1, \dots, J$ . Then the following properties are equivalent:

- (i)  $M(y_0, s_0) \subset \bigcup_{j=1}^J M(\Gamma_j, h_j)$ .
- (ii) For all  $\epsilon > 0$  there is a neighborhood  $\Gamma_0$  of  $y_0$  such that

$$M(\Gamma_0, s_0) \subset \bigcup_{j=1}^J M(\Gamma_j, h_j + \epsilon).$$

If  $y_0 \in \partial M$  then we may take  $\Gamma_0 \subset \partial M$  in (ii). Moreover, the following properties are equivalent:

- (i')  $M(y_0, s_0) \subset M(y_1, s_1) \cup \bigcup_{j=2}^J M(\Gamma_j, h_j)$ .
- (ii') For all neighborhoods  $\Gamma_1$  of  $y_1$  we have

$$M(y_0, s_0) \subset M(\Gamma_1, s_1) \cup \bigcup_{j=2}^J M(\Gamma_j, h_j).$$

If  $y_1 \in \partial M$  then we may take  $\Gamma_1 \subset \partial M$  in (ii').  $\square$



**4.2. From relations between domains of influences to distance functions.** In this section we prove that the distances

$$(20) \quad d(\gamma(s; y, \nu), z), \quad (s, y) \in \mathcal{N}_{\mathcal{R}}, \quad z \in \mathcal{R},$$

are determined by  $\sigma_{\mathcal{R}}$  and the relation (18). Moreover, we prove an analogous result for the relation (19). To formulate the result, let us recall that we have defined for open  $\Gamma \subset \partial M$  and  $y \in \Gamma$ ,

$$\sigma_{\Gamma}(y) := \sup\{s \in (0, \tau_M(y, \nu)]; \quad d(\gamma(s; y, \nu), \Gamma) = s\}.$$

For open  $\Gamma \subset M^{\text{int}}$  with smooth boundary and  $y \in \partial\Gamma$  we define  $\sigma_{\Gamma}(y)$  by the same formula where  $\nu$  is now the interior unit normal vector of  $M \setminus \Gamma$ . We will show the following lemma.

**Lemma 7.** *Suppose that one of the following is satisfied.*

- (a)  $\Gamma \subset \partial M$  is open and  $y_0 \in \Gamma$ .
- (b)  $\Gamma \subset M^{\text{int}}$  is open with smooth boundary and  $y_0 \in \partial\Gamma$ .

Let  $y_1 \in M$ ,  $t > 0$  and let  $0 < s \leq \sigma_{\Gamma}(y_0)$ . Then the following properties are equivalent:

- (i)  $d(\gamma(s; y_0, \nu), y_1) \leq t$ .
- (ii) For all  $\epsilon > 0$  there is  $\delta > 0$  such that

$$M(y_0, s) \subset M(\Gamma, s - \delta) \cup M(y_1, t + \epsilon).$$

In Section 4.4 we will reconstruct the function  $\sigma_{\mathcal{R}}$ . To this end we will consider here also the modified distance functions  $d_h$ . Let  $\Gamma \subset \partial M$  be open and let  $h \in C^1(\bar{\Gamma})$  satisfy

$$(21) \quad |\text{grad}_{\partial M} h(y)|_g < 1 \quad \text{for all } y \in \bar{\Gamma}.$$

We define the modified normal vector,

$$V(h) := (1 - |\text{grad}_{\partial M} h|_g^2)^{1/2} \nu - \text{grad}_{\partial M} h,$$

and the modified distance to a cut point

$$\sigma_{\Gamma}(y; h) := \sup\{s \in (0, \tau_M(y, V(h))]; \quad d_h(\gamma(s; y, V(h)), \Gamma) = s\}.$$

Notice that  $\sigma_{\Gamma}(y) = \sigma_{\Gamma}(y; 0)$ .

To unify the notation we define  $V(h)$  and  $\sigma_{\Gamma}(y; h)$  also for open  $\Gamma \subset M^{\text{int}}$  with smooth boundary,  $y \in \partial\Gamma$  and  $h = 0$  by the same formulas. That is  $V(h) = \nu$  and  $\sigma_{\Gamma}(y; h) = \sigma_{\Gamma}(y)$  where  $\nu$  is now the interior unit normal vector of  $M \setminus \Gamma$ .

**Lemma 8.** *Let  $\Gamma \subset \partial M$  be open,  $h \in C^1(\bar{\Gamma})$ ,  $x \in M^{\text{int}}$  and suppose that  $y_0 \in \Gamma$  satisfies*

$$d_h(x, y_0) = d_h(x, \Gamma).$$

Moreover, let  $\gamma$  be a unit speed shortest path from  $y_0$  to  $x$ . If  $h$  satisfies (21) then there is  $\rho > 0$  such that  $\gamma((0, \rho)) \subset M^{\text{int}}$  and  $\gamma|_{[0, \rho]}$  is the geodesic with initial velocity  $V(h)$ .

By [1] every shortest path is  $C^1$ , whence we lose no generality with the assumption that  $\gamma$  has unit speed.

*Proof.* Let us denote  $t := d(x, y_0)$ . We prove the existence of  $\rho$  by a contradiction, so suppose that there is a strictly decreasing sequence  $(s_j)_{j=1}^\infty$  in  $(0, t)$  converging to zero such that  $\gamma(s_j) \in \partial M$ . Let us consider boundary normal coordinates  $(r, z) \in [0, \infty) \times \partial M$  in a neighborhood of  $y_0$ . In these coordinates the metric tensor has the form

$$(22) \quad g(r, z) = dr^2 + g^0(r, z)dz^2 = dr^2 + \sum_{j,k=1}^{n-1} g_{jk}^0(r, z)dz^j dz^k.$$

We denote the boundary normal coordinates of  $\gamma(s)$  by  $(r(s), z(s))$ .

Notice that for all  $\epsilon > 0$  there is  $\delta > 0$  such that for all  $s \in [0, \delta]$

$$|\dot{z}(s)|_{g^0(0, z(s))} \leq |\dot{z}(s)|_{g^0(\gamma(s))} + \epsilon \leq |\dot{\gamma}(s)|_{g(\gamma(s))} + \epsilon = 1 + \epsilon.$$

Indeed, the first inequality follows from  $r(0) = 0$  and smoothness of  $g^0$ , and the second one from (22). We denote  $y_j := \gamma(s_j)$  and  $h' := \text{grad}_{\partial M} h$ . Then for small  $\epsilon > 0$ , large  $j$  and  $k > j$ ,

$$\begin{aligned} |h(y_j) - h(y_k)| &= \left| \int_{s_k}^{s_j} \partial_s h(z(s)) ds \right| \leq \|h'\|_{C(\bar{\Gamma})} \int_{s_k}^{s_j} |\dot{z}(s)|_{g^0(0, z(s))} ds \\ &= \|h'\|_{C(\bar{\Gamma})} (1 + \epsilon)(s_j - s_k) < s_j - s_k. \end{aligned}$$

By taking the limit  $k \rightarrow \infty$  we have  $|h(y_j) - h(y_0)| < s_j$ . Hence for large  $j$

$$\begin{aligned} d_h(x, y_j) &\leq l(\gamma|_{[s_j, t]}) - h(y_j) < t - s_j - h(y_0) + s_j \\ &= d_h(x, y_0) = d_h(x, \Gamma), \end{aligned}$$

which is a contradiction as  $y_j \in \Gamma$  for large  $j$ . We have shown that there is  $\rho > 0$  such that  $\gamma((0, \rho)) \subset M^{\text{int}}$ . In particular  $\gamma|_{[0, \rho]}$  coincides with a geodesic. The fact  $\dot{\gamma}(0) = V(h)$  follows from a variation argument, see e.g. [36, p. 99] for a similar proof.  $\square$

**Lemma 9.** *Let us suppose one of the following*

- (a)  $\Gamma \subset \partial M$  is open,  $y \in \Gamma$  and  $h \in C(\bar{\Gamma})$  satisfies  $h(y) = 0$ .
- (b)  $\Gamma \subset M^{\text{int}}$  is open and has smooth boundary,  $y \in \partial\Gamma$  and  $h = 0$  identically.

*Let  $t > 0$ . Then the following properties are equivalent*

- (i) *There is  $x \in M$  such that  $d(x, y) = d_h(x, \Gamma) = t$ .*

(ii) For all  $s < t$ ,  $M(y, t) \not\subset M(\Gamma, s + h)$ .

Moreover, if  $h \in C^1(\bar{\Gamma})$  satisfies (21) and  $t \leq \tau_M(y, V(h))$ , then  $x$  in (i) is unique and  $x = \gamma(t; y, V(h))$ .

*Proof.* It is clear that (i) implies (ii). Let us show (i) assuming (ii). We choose a sequence  $(x_j, s_j)_{j=1}^\infty$  in  $M(y, t) \times (0, t)$  such that  $x_j \notin M(\Gamma, s_j + h)$  and  $s_j \rightarrow t$ . By considering a subsequence we may assume that  $x_j \rightarrow x$ . As  $M(y, t)$  is closed we have  $x \in M(y, t)$ , whence

$$d_h(x, y) = d(x, y) \leq t.$$

Moreover,  $x_j \notin M(\Gamma, s_j + h)$  implies

$$d_h(x, y) \geq d_h(x, \Gamma) = \lim_{j \rightarrow \infty} d_h(x_j, \Gamma) \geq \lim_{j \rightarrow \infty} s_j = t.$$

We have shown (i).

Let us proceed to show the uniqueness. Let  $\gamma$  be a unit speed shortest path from  $y$  to  $x$ . Then  $\gamma$  coincides with  $\gamma(\cdot; y, V(h))$  as long as it does not intersect  $\partial M$ . Indeed, this follows from Lemma 8 in the case (a) and from an analogous variation argument in the case (b). Finally,  $d(x, y) = t \leq \tau_M(y, V(h))$  implies that  $x = \gamma(t; y, V(h))$ .  $\square$

**Lemma 10.** *Let us suppose one of the following*

- (a)  $\Gamma \subset \partial M$  is open,  $y \in \Gamma$  and  $h \in C^1(\bar{\Gamma})$  satisfies  $h(y) = 0$  and (21).
- (b)  $\Gamma \subset M^{\text{int}}$  is open and has smooth boundary,  $y \in \partial\Gamma$  and  $h = 0$  identically.

Let  $r > 0$ . Then (i) implies (ii) where

- (i)  $r \leq \sigma_\Gamma(y; h)$
- (ii) For all  $t \in (0, r]$  and  $s \in (0, t)$ ,  $M(y, t) \not\subset M(\Gamma, s + h)$ .

Moreover, if  $\gamma(\cdot; y, V(h))$  is transversal to  $\partial M$  then (ii) implies (i).

*Proof.* If (i) holds then  $x = \gamma(t; y, V(h))$  satisfies (i) of Lemma 9 for all  $t \in (0, r]$ , whence (ii) holds.

Let us now suppose that  $\gamma(\cdot; y, V(h))$  is transversal to  $\partial M$ . We will first prove that  $r > \tau_M(y, V(h))$  together with (ii) yield a contradiction. Let us denote  $\tau := \tau_M(y, V(h))$  and let  $t \in (\tau, r)$ . By Lemma 9 there is  $x \in M$  satisfying  $d(x, y) = d_h(x, \Gamma) = t$ . Thus any shortest path  $\gamma$  from  $y$  to  $x$  coincides with  $\gamma(\cdot; y, V(h))$  on the interval  $[0, \tau]$ . By transversality  $\dot{\gamma}(\tau) \notin T\partial M$ , whence  $\gamma$  is not  $C^1$  at  $\tau \in (0, t)$ . This is a contradiction, since  $\gamma : [0, t] \rightarrow M$  is a shortest path, whence it is  $C^1$ , see [1]. We have shown that (ii) implies  $r \leq \tau_M(y, V(h))$ .

Now we see that (ii) implies also (i) by applying Lemma 9 for all  $t \in (0, r]$ .  $\square$

*Proof of Lemma 7.* We denote  $x_0 := \gamma(s; y_0, \nu)$ . By Lemmas 10 and 9 the point  $x_0$  is the only point  $x \in M$  satisfying  $d(x, y_0) = d(x, \Gamma) = s$ . In particular,  $x_0 \notin M(\Gamma, s - \delta)$  for  $\delta > 0$ .

If (ii) holds, then  $x_0 \in M(y_1, t + \epsilon)$  for all  $\epsilon > 0$ . Hence  $d(x_0, y_1) \leq t + \epsilon$  for all  $\epsilon > 0$ , and we have (i).

Let us now assume (i) and let  $\epsilon > 0$  and  $x \in M(y_0, s)$ . If there does not exist  $\delta > 0$  such that  $x \in M(\Gamma, s - \delta)$ , then  $d(x, \Gamma) > s - \delta$  for all  $\delta > 0$ . Thus

$$s \geq d(x, y_0) \geq d(x, \Gamma) \geq s,$$

and  $x = x_0 \in M(y_1, t + \epsilon)$ . We have shown (ii).  $\square$

**4.3. From distance funtions to local reconstructions of the manifold.** By Lemma 7 the distances,

$$(23) \quad d(\gamma(s; y, \nu), z), \quad (s, y) \in \mathcal{N}_{\mathcal{R}}, \quad z \in \mathcal{R},$$

are determined by  $\sigma_{\mathcal{R}}$  and the relation (18). The considerations in [28, Section 4.4.6] imply that the distances (23) determine  $(M_{\mathcal{R}}, g)$  in the boundary normal coordinates (6). The reconstruction of  $(M_B, g)$  from the relation (19) is rather similar. However, we will describe it here for the sake of completeness.

Let  $x \in M$  and  $\rho > 0$  and denote  $B := M(x, \rho)$ . Let us suppose that  $\rho$  is small enough so that  $B$  is contained in a normal neighborhood of  $x$  in  $M^{\text{int}}$  and consider the interior data  $\Lambda_{\mathcal{S}, B}$ .

We let  $r \in (0, \rho)$  and denote

$$Y_r(\xi) := \gamma(r; x, \xi), \quad Y_r : S_x M \rightarrow \partial M(x, r).$$

Then  $Y_r$  is a diffeomorphism and  $\gamma(s; Y_r(\xi), \nu) = \gamma(s + r; x, \xi)$ . Notice also that

$$(24) \quad r + \min_{y \in \partial M(x, r)} \sigma_{M(x, r)}(y) = \sigma^B.$$

Lemma 7 implies that  $Y_r$ ,  $\sigma^B$  and the relation (19) for  $B = M(x, r)$  determine the distances,

$$d(\gamma(s + r; x, \xi_0), \gamma(r; x, \xi_1)), \quad \xi_0, \xi_1 \in S_x M, \quad 0 < s < \sigma^B - r.$$

Let us denote by  $\text{dist}$  the distance function  $d$  in the geodesic normal coordinates (7). We have shown that  $(B, g)$ ,  $\sigma^B$  and  $\Lambda_{\mathcal{S}, B}$  determine the distances,

$$(25) \quad \text{dist}((s, \eta), (r, \xi)), \quad (s, \eta) \in \mathcal{N}_B, \quad (r, \xi) \in (0, \rho) \times S_x M.$$

**Lemma 11.** *Let  $(s_0, \eta_0) \in \mathcal{N}_B$  and let us consider the differentiated distance function*

$$\Phi(r, \xi) := d_{(s, \eta)} \text{dist}((s, \eta), (r, \xi))|_{(s, \eta) = (s_0, \eta_0)}, \quad (r, \xi) \in (0, \rho) \times S_x M.$$

*Then there is  $r_0 > 0$  such that the image of  $(0, r) \times S_x M$  under  $\Phi$  is open in  $S_{(s_0, \eta_0)}^* \mathcal{N}_B$  for all  $0 < r < r_0$ .*

*Proof.* For  $y \in M$ ,  $\eta \in S_y M$  and  $t > 0$  we denote  $\exp_y(t\eta) := \gamma(t; y, \eta)$ . As  $(s_0, \eta_0)$  is in a normal coordinate neighborhood of  $x = 0$  the same is true for  $(r, \xi)$  with  $r$  small enough. Thus

$$\Phi^\sharp(r, \xi) := \text{grad}_{(s, \eta)} \text{dist}((s, \eta), (r, \xi))|_{(s, \eta) = (s_0, \eta_0)} = -P \exp_{(s_0, \eta_0)}^{-1}(r, \xi),$$

where  $P$  is the projection,

$$Pv := \frac{v}{|v|_g}, \quad P : T_{(s_0, \eta_0)} \mathcal{N}_B \rightarrow S_{(s_0, \eta_0)} \mathcal{N}_B.$$

As  $\exp_{(s_0, \eta_0)}^{-1}$  is a local diffeomorphism around 0 and  $P$  is an open map, we see that  $\Phi^\sharp((0, r) \times S_x M)$  is open in  $S_{(s_0, \eta_0)} \mathcal{N}_B$  for small enough  $r$ . The claim follows by using the isomorphism  $S_{(s_0, \eta_0)} \mathcal{N}_B \rightarrow S_{(s_0, \eta_0)}^* \mathcal{N}_B$  induced by the metric  $g$ .  $\square$

Lemma 11 implies that the second order homogeneous polynomial  $g(s_0, \eta_0)$  is determined on the space  $T_{(s_0, \eta_0)}^* \mathcal{N}_B = \mathbb{R}^n$  by the distances (25). Thus  $g$  is determined also on  $T\mathcal{N}_B$ . To summarize, when we are given  $\Lambda_{\mathcal{S}, B}$ ,  $\sigma^B$  and  $(B, g)$  we can determine  $(M_B, g)$  in the geodesic normal coordinates (7).

**4.4. Reconstruction of distances to cut points using modified distance functions.** In this section we show that for open  $\Gamma \subset \partial M$  the distance to a cut point  $\sigma_\Gamma$  is determined by the relation

$$\{(y, t, h) \in \Gamma \times (0, \infty) \times C^1(\bar{\Gamma}); M(y, t) \subset M(\Gamma, h)\}.$$

Moreover, we show that for a small ball  $B \subset M^{\text{int}}$  the cut time  $\sigma^B$  is determined by the relation

$$\{(y, t, s) \in \Gamma \times (0, \infty)^2; M(y, t) \subset M(B, s)\}.$$

**Lemma 12.** *Let  $\Gamma \subset \partial M$  be open and let  $y_0 \in \Gamma$ . Then the map*

$$\sigma_\Gamma : \Gamma \times C^1(\bar{\Gamma}) \rightarrow \mathbb{R}$$

*is lower semicontinuous at  $(y_0; 0)$ . Moreover, if  $\sigma_\Gamma(y_0; 0) < \tau_M(y_0, \nu)$  then  $\sigma_\Gamma$  is continuous at  $(y_0; 0)$ .*

*Proof.* We prove the semicontinuity by a contradiction, so suppose that there is a sequence  $((y_j, h_j))_{j=1}^\infty$  converging to  $(y_0, 0)$  such that  $\liminf_{j \rightarrow \infty} \sigma_\Gamma(y_j; h_j) < \sigma_\Gamma(y_0; 0)$ . We denote  $h_0 = 0$  and

$$\sigma_j := \sigma_\Gamma(y_j; h_j), \quad \tau_j := \tau_M(y_j, V(h_j)), \quad \text{for } j \geq 0.$$

As  $[0, \sigma_0]$  is compact, we may consider a subsequence and assume that  $\sigma_j \rightarrow \sigma_\infty$ . We let  $T \in (\sigma_\infty, \sigma_0)$  and define

$$x_j := \gamma(T; y_j, V(h_j)), \quad t_j := d_{h_j}(x_j, \Gamma).$$

Notice that  $x_j$  is well defined for large  $j$ . Indeed, the exit time function  $\tau_M$  is lower semicontinuous, see e.g. [18], whence

$$\liminf_{j \rightarrow \infty} \tau_j \geq \tau_0 \geq \sigma_0 > T.$$

We denote  $x_0 := \gamma(T; y_0, \nu)$ . Then continuity properties of the modified distances imply  $t_j \rightarrow d(x_0, \Gamma)$ . Moreover,  $T > \sigma_\infty$  implies  $t_j < T$  for large  $j$ , and  $T < \sigma_0$  implies  $d(x_0, \Gamma) = T$ . To summarize, we may consider a subsequence and assume that

$$(26) \quad t_j < \lim_{j \rightarrow \infty} t_j = d(x_0, \Gamma) = T < \tau_j.$$

There is  $z_j \in \bar{\Gamma}$  such that  $d_{h_j}(x_j, z_j) = t_j$ . By considering a subsequence we may assume that  $z_j \rightarrow z_\infty \in \bar{\Gamma}$ . We see that  $z_\infty$  is a closest point to  $x_0$  in  $\bar{\Gamma}$  since

$$d(x_0, z_\infty) = \lim_{j \rightarrow \infty} d_{h_j}(x_j, z_j) = \lim_{j \rightarrow \infty} t_j = T.$$

However,  $T < \sigma_0$  implies that  $z_\infty = y_0$ , see e.g. [14, pp. 144, 115]. By considering a subsequence we may assume that  $z_j \in \Gamma$  since  $z_j \rightarrow y_0 \in \Gamma$ . Lemma 8 and the inequality (26) imply that

$$x_j = \gamma(t_j; z_j, V(h_j)).$$

As  $T < \sigma_0$ , the map  $(r, y) \mapsto \gamma(r; y, \nu)$  is a local diffeomorphism at  $(T, y_0) \in (0, \infty) \times \Gamma$ , see e.g. [14, p. 144, Th. III.2.2]. Moreover, the map

$$\begin{aligned} \alpha : C^1(\bar{\Gamma}) \times (0, \infty) \times \Gamma &\rightarrow C^1(\bar{\Gamma}) \times M, \\ \alpha(h, r, y) &:= (h, \gamma(r; y, V(h))) \end{aligned}$$

is a local diffeomorphism at  $(0, T, y_0)$  since its derivative is of the form

$$\begin{pmatrix} Id & 0 \\ A & d_{(r,y)}\gamma(r; y, \nu)|_{r=T, y=y_0} \end{pmatrix},$$

where  $A : C^1(\bar{\Gamma}) \rightarrow T_{x_0}M$  is a continuous linear operator. In particular, there is a local inverse  $\beta$  such that in a neighborhood of  $(0, T, y_0)$  we have  $\beta(h, \gamma(r; y, V(h))) = (r, y)$ . Hence for large  $j$

$$\begin{aligned} (t_j, z_j) &= \beta(h_j, \gamma(t_j; z_j, V(h_j))) = \beta(h_j, x_j) = \beta(h_j, \gamma(T; y_j, V(h_j))) \\ &= (T, y_j), \end{aligned}$$

which is a contradiction with (26). We have shown that  $\sigma_\Gamma$  is lower semicontinuous at  $(y_0; 0)$ .

Let us now suppose that  $\sigma_0 < \tau_0$  and show upper semicontinuity by a contradiction. To that end, we suppose that  $\sigma_\infty > \sigma_0$ . We let  $\epsilon > 0$  satisfy  $\sigma_0 + \epsilon < \min(\sigma_\infty, \tau_0)$  and denote  $x_j(\epsilon) = \gamma(\sigma_0 + \epsilon; y_j, V(h_j))$ . Then for large  $j$

$$\sigma_0 + \epsilon = d_{h_j}(x_j(\epsilon), \Gamma).$$

In particular, for small  $\epsilon > 0$

$$\sigma_0 + \epsilon = \lim_{j \rightarrow \infty} d_{h_j}(x_j(\epsilon), \Gamma) = d(\gamma(\sigma_0 + \epsilon; y_0, \nu), \Gamma),$$

which is a contradiction with the definition of  $\sigma_0$ .  $\square$

**Lemma 13.** *Let  $\Gamma \subset \partial M$  be open and let  $y_0 \in \Gamma$ . Then there is  $(y_j, h_j)_{j=1}^\infty \subset \Gamma \times C^1(\bar{\Gamma})$  converging to  $(y_0, 0)$  such that the geodesic  $s \mapsto \gamma(s; y_j, V(h_j))$  is transversal to  $\partial M$ ,  $h_j(y_j) = 0$  and*

$$\lim_{j \rightarrow \infty} \sigma_\Gamma(y_j; h_j) = \sigma_\Gamma(y_0; 0).$$

*In particular,*

$$\liminf_{(y, h) \rightarrow (y_0, 0)} \sigma_\Gamma(y; h) = \sigma_\Gamma(y_0; 0).$$

*Proof.* By [18, Lem. 12] there is a sequence  $(y_j, \eta_j)_{j=1}^\infty \subset \partial_- SM$  converging to  $(y_0, \nu)$  such that  $\gamma(\cdot; y_j, \eta_j)$  is transversal to  $\partial M$  and  $\tau_M(y_j, \eta_j)$  converges to  $\tau_M(y_0, \nu)$  as  $j \rightarrow \infty$ . We may choose  $(h_j)_{j=1}^\infty \subset C^1(\bar{\Gamma})$  converging to zero such that

$$h_j(y_j) = 0 \quad \text{and} \quad \text{grad}_{\partial M} h_j(y_j) = \eta_j|_{T^* \partial M}.$$

In the case when  $\sigma_\Gamma(y_0; 0) < \tau_M(y_0, \nu)$ , the claim follows immediately from Lemma 12. Let us consider the case  $\sigma_\Gamma(y_0; 0) = \tau_M(y_0, \nu)$ . Then

$$\begin{aligned} \sigma_\Gamma(y_0; 0) &= \tau_M(y_0, \nu) = \liminf_{j \rightarrow \infty} \tau_M(y_j, V(h_j)) \geq \liminf_{j \rightarrow \infty} \sigma_\Gamma(y_j; h_j) \\ &\geq \sigma_\Gamma(y_0; 0). \end{aligned}$$

Moreover, by considering a subsequence we may assume that  $\sigma_\Gamma(y_j; h_j)$  converges to  $\liminf_{j \rightarrow \infty} \sigma_\Gamma(y_j; h_j)$  as  $j \rightarrow \infty$ . The second claim follows from the first claim and the lower semicontinuity of  $\sigma_\Gamma$  at  $(y_0; 0)$ .  $\square$

For open  $\Gamma \subset \partial M$ ,  $y \in \Gamma$  and  $h \in C(\bar{\Gamma})$  we define

$$\tilde{\sigma}_\Gamma(y; h) := \sup\{t \in (0, \infty); t \text{ satisfies (ii) of Lemma 9}\}.$$

**Lemma 14.** *Let  $\Gamma \subset \partial M$  be open and  $y \in \Gamma$ . Then*

$$\liminf_{(y,h) \rightarrow (y_0,0)} \tilde{\sigma}_\Gamma(y; h) = \sigma_\Gamma(y_0; 0),$$

where the  $\liminf$  is taken over all  $(y, h) \in \Gamma \times C^1(\bar{\Gamma})$  such that  $h(y) = 0$ .

*Proof.* Lemmas 10 and 13 imply

$$\liminf_{(y,h) \rightarrow (y_0,0)} \tilde{\sigma}_\Gamma(y; h) \geq \liminf_{(y,h) \rightarrow (y_0,0)} \sigma_\Gamma(y; h) = \sigma_\Gamma(y_0; 0).$$

Let  $(h_j)_{j=1}^\infty$  be as in Lemma 13. Then by Lemma 10

$$\liminf_{(y,h) \rightarrow (y_0,0)} \tilde{\sigma}_\Gamma(y; h) \leq \liminf_{j \rightarrow \infty} \tilde{\sigma}_\Gamma(y_j; h_j) = \lim_{j \rightarrow \infty} \sigma_\Gamma(y_j; h_j) = \sigma_\Gamma(y_0; 0).$$

□

**Lemma 15.** *Let  $x \in M^{\text{int}}$  and let  $\rho > 0$  be small enough so that  $B := M(x, \rho)$  is contained in a normal neighborhood of  $x$  in  $M^{\text{int}}$ . For  $r > 0$  the following properties are equivalent:*

- (i)  $r + \rho \leq \sigma^B$ .
- (ii) For all  $t \in (0, r]$ ,  $s \in (0, t)$  and  $y \in \partial B$ ,  $M(y, t) \not\subset M(B, s)$ .

*Proof.* By Lemma 10 and (24) is enough to show that  $\gamma(\cdot; y, \nu)$  is transversal to  $\partial M$  if  $y \in \partial B$  satisfies  $\tau := \tau_M(y, \nu) = \min_{z \in \partial B} \tau_M(z, \nu)$ . Let  $\xi \in S_x M$  satisfy  $\gamma(\rho; x, \xi) = y$ . Then

$$\gamma(\tau; y, \nu) = \gamma(\tau + \rho; x, \xi)$$

is a closest point to  $y_0$  on  $\partial M$ . Thus  $\gamma(\cdot; y_0, \nu)$  intersects  $\partial M$  normally, in particular, it is transversal to  $\partial M$ . □

Summarizing, when we are given  $\Lambda_{\mathcal{S}, \mathcal{R}}$ , we may use Proposition 2 to first determine  $\sigma_{\mathcal{R}}$  by Lemma 14 and then to reconstruct the subset  $(M_{\mathcal{R}}, g)$  by Lemma 7 and [28, Section 4.4.6]. Analogously, when  $B$  is as in Lemma 15 and we are given  $\Lambda_{\mathcal{S}, B}$  together with  $(B, g)$ , we may use Proposition 2 to first determine  $\sigma^B$  by Lemma 15 and then to reconstruct the subset  $(M_B, g)$  by Lemma 7 and Section 4.3.

## 5. GLOBAL RECONSTRUCTION OF THE MANIFOLD

In the previous section we have shown, under the assumptions of Proposition 2, that  $\Lambda_{\mathcal{S}, \mathcal{R}}$  determines  $(M_{\mathcal{R}}, g)$ . In fact, we have described a method to reconstruct  $(M_{\mathcal{R}}, g)$  in the boundary normal coordinates.



We will show next that  $\Lambda_{\mathcal{S},\mathcal{R}}$  determines the interior data  $\Lambda_{\mathcal{S},B}$  for such a ball  $B \subset M_{\mathcal{R}}$  that there is  $\Gamma \subset \mathcal{R}$  and  $t_0 > 0$  satisfying

$$B \subset M(\Gamma, t_0) \subset M_{\mathcal{R}}.$$

Let  $T > t_0$ . By the finite speed of propagation for the wave equation (1),  $(M_{\mathcal{R}}, g)$  determines  $u^f(T)$  for all  $f \in C_0^\infty((T - t_0, T) \times \Gamma)$ . Moreover, as the functions

$$u^f(T), \quad f \in C_0^\infty((T - t_0, T) \times \Gamma),$$

are dense in  $L^2(M(\Gamma, t_0))$ , we see using Lemma 1 that  $\Lambda_{\mathcal{S},\mathcal{R}}$  determines  $u^\psi(T)|_B$  for  $\psi \in C_0^\infty((t_0, \infty) \times \mathcal{S})$ . By varying  $T > t_0$  and noticing that the equation (1) is invariant with respect to translation in time, we see that  $u^\psi|_{(0,\infty) \times B}$  is determined for  $\psi \in C_0^\infty((0, \infty) \times \mathcal{S})$ . That is,  $\Lambda_{\mathcal{S},\mathcal{R}}$  determines  $\Lambda_{\mathcal{S},B}$ .

Let  $x \in M^{\text{int}}$  and suppose that  $g$  is known in a neighborhood of  $x$ . Then we can choose  $\rho > 0$  small enough so that  $B := M(x, \rho)$  is contained in a known normal neighborhood of  $x$  in  $M^{\text{int}}$ . Let us suppose that  $\Lambda_{\mathcal{S},B}$  is also known. We have seen that  $\Lambda_{\mathcal{S},B}$  determines  $(M_B, g)$ . Let  $B' \subset M_B$  be a ball. Then there is  $t_0 > 0$  such that

$$B' \subset M(B, t_0) \subset M_B.$$

An argument analogous with the argument above shows that  $\Lambda_{\mathcal{S},B}$  determines  $\Lambda_{\mathcal{S},B'}$ .

Let us now show how local reconstructions  $(M_B, g)$  and  $(M_{B'}, g)$  can be glued together. Let  $x_1 \in M_B$  and  $x_2 \in M_{B'}$ . We have seen that the maps  $\Lambda_{\mathcal{S},B}$  and  $\Lambda_{\mathcal{S},B'}$  determine the maps  $\Lambda_{\mathcal{S},B(x_j, \rho)}$ ,  $j = 1, 2$ , for small  $\rho > 0$ . By Lemma (1) we can compute the inner products

$$\begin{aligned} & (u^{f_1-f_2}(T), u^\psi(T))_{L^2(M)} \\ &= (u^{f_1}(T), u^\psi(T))_{L^2(M)} - (u^{f_2}(T), u^\psi(T))_{L^2(M)}, \end{aligned}$$

for  $f_j \in C_0^\infty((0, \infty) \times B(x_j, \rho))$ , and  $\psi \in C_0^\infty((0, \infty) \times \mathcal{S})$ . The set

$$S_\rho := \{s \in (0, \infty); M(B(x_1, \rho), s) \subset M(B(x_2, \rho), s)\}$$

is determined by Proposition 2. Moreover,  $x = y$  if and only if  $S_\rho = (0, \infty)$  for all  $\rho > 0$ . That is, we know how to identify points in  $M_B \cap M_{B'}$ . In particular, we can reconstruct the transition functions, whence we have constructed  $(M_B \cup M_{B'}, g)$ .

Let us consider the collection

$$\begin{aligned} \mathcal{M} &:= \{B; B \subset M^{\text{int}} \text{ is a ball,} \\ &\quad \Lambda_{\mathcal{S},\mathcal{R}} \text{ determines } (M_B, g) \text{ and } \Lambda_{\mathcal{S},B}\}. \end{aligned}$$

We have shown that  $\mathcal{M}$  is nonempty and that

if  $x \in M_B$  and  $B \in \mathcal{M}$  then  $M(x, \rho) \in \mathcal{M}$  for small  $\rho > 0$ .

In particular, the open set

$$U := \bigcup_{B \in \mathcal{M}} M_B \subset M^{\text{int}}$$

is nonempty. We will show next that it is closed in  $M^{\text{int}}$ . Let  $x \in M^{\text{int}}$  and let  $x_j \in U$ ,  $j \in \mathbb{N}$ , converge to  $x \in M^{\text{int}}$ . Then there is a uniformly normal neighborhood  $W$  of  $x$  and  $j \in \mathbb{N}$  such that  $x_j \in W$ , see e.g. [36, Lem. 5.12]. For small  $\rho > 0$ ,  $B' := M(x_j, \rho) \in \mathcal{M}$  since  $x_j \in M_B$  for some  $B \in \mathcal{M}$ . As  $W$  is uniformly normal, we have  $W \subset M_{B'}$ . Hence  $x \in U$  and we have shown that  $U$  is closed in  $M^{\text{int}}$ . As  $M^{\text{int}}$  is assumed to be connected, we have  $U = M^{\text{int}}$ .

We have shown that  $(M^{\text{int}}, g)$  is determined by  $\Lambda_{\mathcal{S}, \mathcal{R}}$ . The smooth Riemannian structure allows us to recover also the closure  $(M, g)$ , see e.g. [26, p. 2116]. This proves Theorem 1.

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UNIVERSITY OF HELSINKI, DEPARTMENT OF MATHEMATICS AND STATISTICS,  
P.O. Box 68 FI-00014